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Comparing Model Update Error Residuals and Effects on Model Predictive Accuracy

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The effect of error residual choice on the predictive capability of an updated structural model is examined. First, analytical expressions are developed that relate errors in dynamically measured static flexibility matrices, stiffness matrices, and modal matrices. The analysis shows that flexibility, stiffness, and modal error residuals correspond to unique weightings on the error in the measured modal data. Next, the flexibility weighting indicator and the stiffness weighting indicator are defined to provide a simple means of ranking individual modes in terms of their potential contribution to errors in a measured static flexibility or stiffness matrix. Finally, expressions are developed for bounding model prediction errors when static flexibility, stiffness, or modal error residuals are used. The results show that the amount of uncertainty in predicted static displacements, static loads, and mode shapes and frequencies is directly related to the choice of error residual and is different in each case. These results are also illustrated using numerical simulations for a modified version of Kabe's problem, which includes model form errors.

Nomenclature

$\{f\}$ = vector of applied forces
G = static flexibility matrix
K = stiffness matrix
M = mass matrix
$\{q\}$ = vector of physical displacements
Δ = perturbation
Λ = eigenvalue matrix
λ_i = ith eigenvalue
μ = modal mass
Φ = modal matrix
$\{\phi_i\}$ = ith mode shape vector
$\| \ \|$ = vector or matrix norm

Superscripts

T = transpose
-1 = matrix inverse

Introduction

IN recent years, there has been a significant amount of research published focusing on methods for reconciling finite element models of structures with data obtained from experimental testing. Most of the techniques attempt to modify stiffness and/or mass matrices such that the mode shapes and frequencies of the model closely match experimentally measured modal parameters. This is typically accomplished by minimizing an error residual that expresses the difference between analytical and experimental modal parameters. A variety of methods have been proposed, and they generally fall within three general classes: optimal matrix update (OMU), sensitivity-based parameter update, and eigenstructure assignment.[1] All of these techniques have successfully demonstrated that updated models may be obtained that have improved correlation to experimentally measured modal parameters.

Recently, other techniques have emerged that attempt to reconcile a model on the basis of static rather than modal behavior. One class of these techniques uses dynamic measurements to form a static flexibility matrix. The model is then correlated to the experimental data by minimizing a matrix error residual expressed in terms of static flexibility matrices.[2–4] Despite the practical difficulties associated with the formation of a high-order stiffness matrix from high-frequency modal data, another alternative might be to express error residuals in terms of stiffness matrices. Stiffness matrices formed from static test data might also be used.

This paper begins to explore the following question: How does the choice of error residual impact the predictive accuracy of the resulting model? To answer this, how static flexibility, stiffness, and modal errors are related to one another is examined. By determining these relationships, comparisons of various model update algorithms can be made through inspection of their error residuals. In addition, these relationships are useful for studying effects due to modal truncation, spatial incompleteness, and measurement noise. Using these relationships, indicators are developed for estimating the relative modal contributions to flexibility and stiffness matrix errors. Finally, numerical simulations are conducted that demonstrate the effect of error residual choice on model predictive accuracy.

Theoretical Development

Relating Static Flexibility and Modal Matrix Errors

To obtain a relationship between the static flexibility and modal errors, we start with the expression for the static flexibility matrix in modal form given by

$$G = \Phi \Lambda^{-1} \Phi^T \tag{1}$$

Adding error to Eq. (1) leads to

$$G + \Delta G = (\Phi + \Delta\Phi)(\Lambda + \Delta\Lambda)^{-1}(\Phi + \Delta\Phi)^T \tag{2}$$

where the Δ terms are errors in the respective matrices. Using the modified matrices formula[5] (matrix inversion lemma), the inverse term may be expanded to get the following:

$$(\Lambda + \Delta\Lambda)^{-1} = \Lambda^{-1} - \Lambda^{-1}(\Lambda^{-1} + \Delta\Lambda^{-1})^{-1}\Lambda^{-1} \tag{3}$$

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Substituting Eq. (3) into Eq. (2) gives us the following matrix equation relating modal errors to the error in the static flexibility matrix:

$$\Delta G = -\Phi\Lambda^{-1}(\Lambda^{-1} + \Delta\Lambda^{-1})^{-1}\Lambda^{-1}\Phi^T + \Phi(\Lambda + \Delta\Lambda)^{-1}\Delta\Phi^T + \Delta\Phi(\Lambda + \Delta\Lambda)^{-1}\Phi^T + \Delta\Phi(\Lambda + \Delta\Lambda)^{-1}\Delta\Phi^T \tag{4}$$

It is interesting to note that the right-hand side (RHS) of Eq. (4) is the modal error residual, which is equivalent to the static flexibility matrix error residual developed by Denoyer and Peterson.[2] Assuming perfect data and model form, this is the modal error residual required in a modal-based update method to duplicate the results of an update that minimizes static flexibility error.

To better understand the relationship between errors in a given mode and errors in the static flexibility matrix, it is desirable to consider the ith-mode contribution to the error in the static flexibility matrix, ΔG_i. Because Λ and $\Delta\Lambda$ are diagonal, Eq. (4) decouples such that

$$\Delta G = \sum_{i=1}^{\text{no. modes}} \Delta G_i \tag{5}$$

where ΔG_i is given by

$$\Delta G_i = \frac{1}{\lambda_i + \Delta\lambda_i}\left(-\frac{\Delta\lambda_i}{\lambda_i}\{\phi_i\}\{\phi_i\}^T + \{\phi_i\}\{\Delta\phi_i\}^T\{\Delta\phi_i\}\{\phi_i\}^T + \{\Delta\phi_i\}\{\Delta\phi_i\}^T\right) \tag{6}$$

and $\Delta\lambda_i$ and $\{\Delta\phi_i\}$ are the errors in the ith eigenvalue and mode shape, respectively. If we assume that $\Delta\lambda_i/\lambda_i$ is small and neglect the second-order term $\{\Delta\phi_i\}\{\Delta\phi_i\}^T$, we obtain the following estimate for ΔG_i:

$$\Delta G_i \approx (1/\lambda_i)\big(\{\phi_i\}\{\Delta\phi_i\}^T + \{\Delta\phi_i\}\{\phi_i\}^T\big) \tag{7}$$

From Eq. (7), we see that the mode shape error is weighted by the mode shape and inversely weighted by frequency. If $\{\Delta\phi_i\}$ were known, Eq. (7) could be used to rank a set of modes based on their contribution to the static flexibility error ΔG_i. Unfortunately, $\{\Delta\phi_i\}$ is generally not known. Later, bounding theorems are used to develop an indicator for ranking modes based on their potential contributions to error in the static flexibility matrix.

Relating Stiffness and Modal Matrix Errors

Deriving the relationship between stiffness error and modal error follows the approach just used. For convenience, it is assumed that the stiffness matrix is nonsingular, and we start with the modal representation of the stiffness matrix found by taking the inverse of Eq. (1),

$$K = (\Phi\Lambda^{-1}\Phi^T)^{-1} \tag{8}$$

Adding error, we get

$$K + \Delta K = [(\Phi + \Delta\Phi)(\Lambda + \Delta\Lambda)^{-1}(\Phi + \Delta\Phi)^T]^{-1} \tag{9}$$

Expanding Eq. (9) and rearranging leads to the following expression for the stiffness error:

$$\Delta K = [\Phi(\Lambda + \Delta\Lambda)^{-1}\Phi^T + \Phi(\Lambda + \Delta\Lambda)^{-1}\Delta\Phi^T + \Delta\Phi(\Lambda + \Delta\Lambda)^{-1}\Phi^T + \Delta\Phi(\Lambda + \Delta\Lambda)^{-1}\Delta\Phi^T]^{-1} - [\Phi\Lambda^{-1}\Phi^T]^{-1} \tag{10}$$

Assuming modal and spatial completeness, the RHS of Eq. (10) gives the modal error residual, which is equivalent to a stiffness matrix error residual. By comparison to Eq. (4), this is the modal error residual required in a modal-based update method to duplicate the results of an update using a stiffness matrix error residual, assuming perfect data and model form.

As was done with flexibility, it would be desirable to decouple Eq. (10) and examine the ith-mode contribution to the stiffness matrix error. Unfortunately, Eq. (10) cannot be decoupled due to the matrix inverses on the bracketed expressions. Therefore, an expression for the ith-mode contribution to the stiffness matrix is unavailable.

Relating Flexibility and Stiffness Matrix Errors

The relationship between static flexibility error and stiffness error is derived from the definition of the static flexibility matrix as the inverse of the stiffness matrix,

$$G \equiv K^{-1} \tag{11}$$

Adding error, Eq. (11) becomes

$$G + \Delta G = (K + \Delta K)^{-1} \tag{12}$$

and the inverse can be expanded as in the preceding sections to get

$$G + \Delta G = K^{-1} - K^{-1}(K^{-1} + \Delta K^{-1})^{-1}K^{-1} \tag{13}$$

which reduces to

$$\Delta G = -G(G + \Delta K^{-1})^{-1}G \tag{14}$$

Alternatively, the inverse of Eq. (11) could be used, resulting in an expression for the error in the stiffness matrix given by the following:

$$\Delta K = -K(K + \Delta G^{-1})^{-1}K \tag{15}$$

Using Eqs. (4), (10), (14), and (15), errors in the static flexibility matrix, stiffness matrix, and modal quantities can be completely related to one another. The consequence is that any error residual written in terms of one of these quantities could be expressed in terms of the others, although the resulting residual may be quite complicated. Although useful for analysis and gaining insight, practical application of these expressions is hampered because only a subset of the modal data is usually available. In addition to modal incompleteness, the data set is often spatially incomplete, with data available only from a small number of degrees of freedom (DOFs). The effect of modal and spatial incompleteness with respect to error residuals involving static flexibility matrices is investigated by Denoyer.[4]

Indicators for Relative Modal Contribution to Flexibility and Stiffness Errors

For any error residual that depends on measured modal data, it is desirable to have the ability to determine how the individual modes impact the error residual. From a practical standpoint, this allows one to judge how sensitive the error residual will be to a particular mode and to determine which modes will produce the most corruption due to measurement noise. In this section, the problem of determining the relative contribution of individual modes to error in a static flexibility matrix or stiffness matrix is examined.

Flexibility Weighting Indicator

If the errors in the individual mode shapes and frequencies are known, Eq. (6) can be used to compute the exact contribution of individual modes to the flexibility matrix error. When the modal errors are unknown, we can attempt to bound the flexibility error due to individual modes. The modes can then be ranked in terms of their potential contribution to flexibility error, given an assumed bound on the modal errors.

To obtain this modal ranking, we can take the norm of both sides of Eq. (7) to get

$$\|\Delta G_i\| \approx (1/\lambda_i)\big\|\{\phi_i\}\{\Delta\phi_i\}^T + \{\Delta\phi_i\}\{\phi_i\}^T\big\| \le (1/\lambda_i)\big(\big\|\{\phi_i\}\{\Delta\phi_i\}^T\big\| + \big\|\{\Delta\phi_i\}\{\phi_i\}^T\big\|\big) \tag{16}$$

Noting that $\|\{\phi_i\}\{\Delta\phi_i\}^T\| = \|\{\Delta\phi_i\}\{\phi_i\}^T\|$, Eq. (16) reduces to the following:

$$\|\Delta G_i\| \le (2/\lambda_i)\big\|\{\phi_i\}\{\Delta\phi_i\}^T\big\| \tag{17}$$

If we use a matrix norm that obeys the submultiplicative property[6] (true for p norms and Frobenius norms) and note that $\|\{\Delta\phi_i\}\| = \|\{\Delta\phi_i\}^T\|$, we may write

$$\|\Delta G_i\| \le (2/\lambda_i)\|\{\phi_i\}\|\|\{\Delta\phi_i\}\| \tag{18}$$

which gives an upper bound on the contribution to the static flexibility matrix error due to mode i. From Eq. (18), we see that this bound is determined by a weighting of the mode shape error by the magnitude of the mass-normalized mode shape and inversely weighted with respect to the square of the frequency. In other words, errors in the static flexibility will be dominated by errors in modes with high amplitude and low frequency. Errors in modes with low amplitude and high frequency should have little effect.

If it is assumed that the unknown modal errors have equal amplitude, Eq. (18) can be used to define a simple indicator for ranking a set of modes based on their potential contribution to static flexibility error. Making use of Eq. (18), the flexibility weighting indicator (FWI) is defined as

$$\mathrm{FWI} \equiv \frac{\|\{\phi_i\}\|_2}{\lambda_i} \tag{19}$$

where $\|\{\phi_i\}\|_2$ is the Euclidean or 2 norm of the mode shape given by the following:

$$\|\phi_i\|_2 = \left(\{\phi_i\}^T\{\phi_i\}\right)^{\frac{1}{2}} \tag{20}$$

The FWI can also be viewed as an indicator of the relative contribution of the modal data in forming the static flexibility matrix. To see this, consider the ith-mode contribution to the static flexibility matrix obtained from Eq. (1),

$$G_i = (1/\lambda_i)\{\phi_i\}\{\phi_i\}^T \tag{21}$$

Taking the 2 norm of both sides gives

$$\|G_i\|_2 = (1/\lambda_i)\left\|\{\phi_i\}\{\phi_i\}^T\right\|_2 \tag{22}$$

Again, using the submultiplicative property and noting that $\|\{\phi_i\}\|_2 = \|\{\phi_i\}^T\|_2$ implies that

$$\|G_i\|_2 \leq \frac{\|\{\phi_i\}\|_2}{\lambda_i}\|\{\phi_i\}\|_2 = \mathrm{FWI} \times \|\{\phi_i\}\|_2 \tag{23}$$

which is simply a weighting of the ith-mode-shape magnitude by the FWI.

The FWI should be thought of as an engineering tool that provides a simple means for ranking a set of modes in terms of their potential contribution to static flexibility matrix error. It is an estimate based on available knowledge, when the modal errors are unknown. If one did have access to these errors, Eq. (6) would be utilized directly.

Stiffness Weighting Indicator

Following the analysis of the preceding section, we would also like to define a simple expression for determining the relative modal contributions to the stiffness matrix error. In other words, a relationship is sought such that

$$\|\Delta K_i\| \sim \mathrm{SWI} \times \|\{\Delta\phi_i\}\| \tag{24}$$

where the stiffness weighting indicator SWI is analogous to the FWI given by Eq. (19). As discussed earlier, the problem is that the inverses in Eq. (10) lead to a nonlinear dependence on the mode shape errors. This means that an expression for the ith-mode contribution cannot be extracted because superposition does not apply in this case. However, logic would suggest that the modal weighting for a stiffness matrix residual would be related to the modal weighting for a static flexibility matrix residual because $K = G^{-1}$.

To gain insight into this problem, consider the single-DOF mass–spring oscillator, shown in Fig. 1 For this problem, the eigenvalue λ is given by

$$\lambda = k/m \tag{25}$$

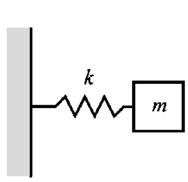

Fig. 1 Single-DOF mass–spring oscillator.

which implies that the stiffness and flexibility are given by the following:

$$k = \lambda m \tag{26}$$

$$g = k^{-1} = 1/\lambda m \tag{27}$$

Because this is a single-DOF system, the mass-normalized mode shape has only one element and is found from

$$\phi m \phi^T = 1 \tag{28}$$

which implies that

$$\phi = 1\big/\sqrt{m} \tag{29}$$

Now, consider the FWI for this problem, given by Eq. (19),

$$\mathrm{FWI} = \frac{\|\phi\|_2}{\lambda} = \frac{\left(1/\sqrt{m}\right)}{\lambda} = \frac{1}{\lambda\sqrt{m}} \tag{30}$$

In the preceding section, it was shown that the magnitude of the ith-mode contribution to the static flexibility matrix is bounded by a weighting of the mass-normalized mode shape magnitude by the FWI. Applying the FWI weighting in this problem yields

$$\mathrm{FWI} \times \phi = \left(1/\lambda\sqrt{m}\right) \times \left(1/\sqrt{m}\right) = 1/\lambda m \tag{31}$$

which is the exact flexibility, given by Eq. (27). Similarly, we would like the SWI to yield the stiffness magnitude when applied to the mass-normalized mode shape. This implies the following:

$$\mathrm{SWI} \times \phi = \mathrm{SWI} \times \left(1/\sqrt{m}\right) = k = \lambda m \tag{32}$$

If we define the SWI as

$$\mathrm{SWI} \equiv \frac{\lambda_i}{\|\phi_i\|_2^3} \tag{33}$$

the weighting for our simple example becomes

$$\mathrm{SWI} \times \phi = \frac{\lambda}{\|\phi\|_2^3} \times \phi = \frac{\lambda}{\left(1/\sqrt{m}\right)^3} \times \frac{1}{\sqrt{m}} = \lambda m \tag{34}$$

and Eq. (32) is satisfied.

The question is, how do these indicators relate to multiple-DOF systems? For multiple-DOF systems, the norm of the mass-normalized ith-mode shape is given by

$$\|\phi_i\|_2 = 1\big/\sqrt{\mu_i} \tag{35}$$

where μ_i is simply the modal mass associated with the ith mode. Substituting Eq. (35) into Eq. (23), we see that the maximum contribution of a mode to the static flexibility matrix error is given by

$$\|G_i\| \leq (1/\mu_i\lambda_i) \tag{36}$$

where the RHS is sometimes referred to as modal flexibility. Therefore, one may interpret the FWI as the weighting that yields modal flexibility when applied to the mode shape magnitude.

Similarly, when we apply the SWI of Eq. (33) to the mode shape magnitude, we obtain

$$\mathrm{SWI} \times \|\phi_i\|_2 = \frac{\lambda_i}{\|\phi_i\|_2^3} \times \|\phi_i\|_2 = \frac{\lambda_i}{\|\phi_i\|_2^2} = \mu_i\lambda_i \tag{37}$$

where the product of modal mass and frequency is sometimes referred to as modal stiffness. Therefore, an interpretation of the SWI is the weighting that produces modal stiffness when applied to the mode shape magnitude. Because the true ith-mode contribution of the stiffness matrix error cannot be determined, the SWI is used to instead rank the modes based on a weighting of the relative mode shape error by modal stiffness. The assumption in doing this is that errors in modes corresponding to high modal stiffness will also have the largest contribution to errors in the stiffness matrix.

Using the FWI and SWI, it is possible to estimate a priori those modes that have the largest potential contribution to static flexibility or stiffness matrix errors. In the sections that follow, it is demonstrated that choosing the appropriate error residual is necessary to limit errors in either static displacements, static loads, or mode shapes and frequencies. These indicators allow one to calculate those modes that will be important in obtaining accurate values for these predicted quantities.

For the purpose of this analysis, it has been assumed that the error residual is written in terms of the same DOF as the measured data. In cases where modal reduction or eigenvector expansion is used, additional errors would be introduced. However, one might expect these errors to be small relative to the modal errors themselves. In addition, it is possible to write some error residuals exactly in terms of the reduced DOF set. This is true of the static flexibility error residuals described by Denoyer and Peterson.[2,3]

Another important factor to consider is the existence of rigid-body modes. For the case of unconstrained structures, it is assumed that the comparison between model and measurement will be based on flexible modes only. In this case, the FWI and SWI may be used in the same manner. However, the theoretical development must be interpreted slightly differently. Instead of considering errors to the flexibility and stiffness matrices, one can use a similar argument that considers only the flexible mode contribution to the flexibility and stiffness matrices. In other words, the rigid-body component is subtracted a priori. This was the approach used to develop the error residual in Denoyer and Peterson.[3]

Applying Weighting Indicators to Kabe's Problem[7]

To illustrate the developed error relationships, we use the eight-DOF example presented by Kabe.[7] Two sets of stiffness values are given for this problem and are shown in Table 1. Using these values, both an exact and a nominal analytical model are generated. The error in the stiffness matrix, ΔK, is taken to be the difference between the stiffness matrix of each model. Similarly, the error in the static flexibility matrix, ΔG, is the difference between the inverses of these two stiffness matrices. Finally, a set of mass-normalized mode shapes and frequencies is generated for each model. The error in the modal matrices, $\Delta\Phi$ and $\Delta\Lambda$, is calculated as the difference between the modal matrices for each model. For this problem, these error matrices are given by the following expressions:

Table 1 Mass and stiffness parameters for Kabe's problem

Mass	Mass value	Spring	Exact stiffness values	Model stiffness values	Stiffness error, %
m_1	0.001	k_1	1.5	2.0	33.3
m_2	1.000	k_2	10.0	10.0	0.0
m_3	1.000	k_3	100.0	200.0	100.0
m_4	1.000	k_4	100.0	200.0	100.0
m_5	1.000	k_5	100.0	200.0	100.0
m_6	1.000	k_6	10.0	10.0	0.0
m_7	1.000	k_7	2.0	4.0	100.0
m_8	0.002	k_8	1.5	2.0	33.3
——	——	k_9	1000.0	1500.0	50.0
——	——	k_{10}	900.0	450.0	−50.0
——	——	k_{11}	1000.0	1500.0	50.0
——	——	k_{12}	1000.0	1500.0	50.0
——	——	k_{13}	900.0	450.0	−50.0
——	——	k_{14}	1000.0	1500.0	50.0

$$\Delta K = K_{\text{exact}} - K_{\text{model}} = \begin{bmatrix} -0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & -500.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -600.0 & 0 & 100.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 250.0 & 100.0 & 100.0 & 0 & 0 \\ 0 & 0 & 100.0 & 100.0 & 250.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100.0 & 0 & -602.0 & 0 & 2.0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -500.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 2.0 & 0.5 & -2.5 \end{bmatrix} \tag{38}$$

$$\Delta G = G_{\text{exact}} - G_{\text{model}} = 10^4 \begin{bmatrix} 1669.9 & 3.3 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 3.3 & 3.3 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.1 & 3.1 & -0.3 & -0.7 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -0.3 & -3.6 & -2.3 & -0.7 & 0.0 & -0.5 \\ 0.0 & 0.0 & -0.7 & -2.3 & -3.6 & -0.3 & 0.0 & -0.2 \\ 0.0 & 0.0 & 0.0 & -0.7 & -0.3 & 3.1 & 0.1 & 1.2 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 3.3 & 2.1 \\ 0.0 & 0.0 & 0.0 & -0.5 & -0.2 & 1.2 & 2.1 & 1191.9 \end{bmatrix} \tag{39}$$

$$\Delta\Phi = \Phi_{\text{exact}} - \Phi_{\text{model}} = \begin{bmatrix} 0.146 & 2.784 & -5.090 & -0.490 & -0.040 & 0.132 & 62.845 & 0.000 \\ 0.055 & 0.919 & -1.350 & -0.147 & 0.011 & 0.012 & -0.222 & -0.000 \\ 0.486 & 0.317 & 0.035 & 0.579 & 0.052 & -0.329 & 0.007 & 0.000 \\ 1.301 & -0.769 & 0.024 & -0.394 & 0.225 & 0.409 & 0.000 & 0.005 \\ 1.300 & 0.722 & 0.120 & 0.379 & 0.239 & -0.399 & -0.002 & -0.001 \\ 0.489 & -0.314 & -0.086 & -0.526 & 0.121 & 0.262 & -0.000 & -0.001 \\ 0.063 & -0.341 & -0.924 & 1.185 & 0.007 & -0.014 & 0.000 & -0.015 \\ 0.612 & -0.698 & -1.049 & 0.116 & 0.144 & 0.261 & -0.000 & -0.014 \end{bmatrix} \tag{40}$$

$$\Delta\Lambda = \Lambda_{\text{exact}} - \Lambda_{\text{model}} = 10^3 \begin{bmatrix} 0.327 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.012 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.495 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.463 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.577 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.501 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.504 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.253 \end{bmatrix} \tag{41}$$

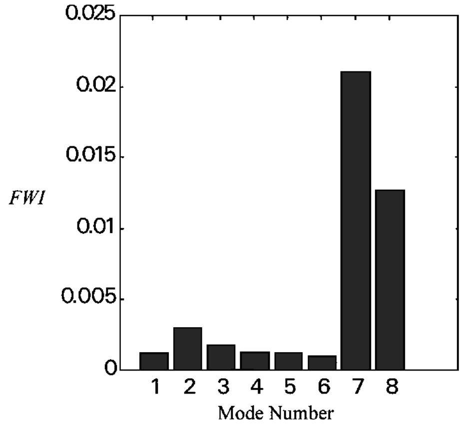

Fig. 2 FWI for mass-normalized modes of Kabe's problem.[7]

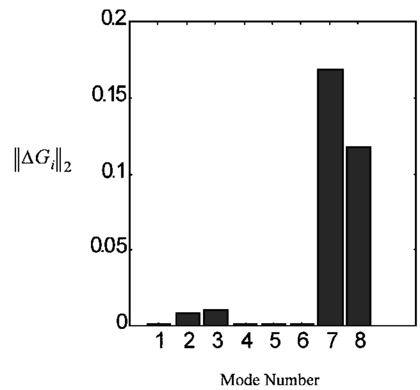

Fig. 3 Norm of *i*th mode contribution to static flexibility matrix error.

Using these quantities, the matrix equations (4), (10), (14), and (15) can all be verified by substitution. By summing over all of the modes in the model, Eqs. (5) and (6) can be verified as well.

To demonstrate the use of the FWI as a viable indicator for the modal contribution to error in the static flexibility matrix, a comparison is made between the value of the FWI for each mode and the 2 norm of the actual ith-mode static flexibility error matrix, given in Eq. (6). The values of the FWI for each mode are plotted in Fig. 2, and the norms of the ith-mode static flexibility error matrices are plotted in Fig. 3. This comparison shows that the FWI correctly predicts the modes that contribute the most to the static flexibility matrix error. In this case, the dominant modes are 7 and 8. If one assumed that the lowest-frequency modes were the greatest contributors, these would instead be the least weighted modes. It is clear from this example that the magnitude of the mass-normalized mode shape plays an important role, as well as the frequency. This dependence is captured by the FWI.

We can also plot the SWI values for this problem (Fig. 4). Note that the dominant modes with respect to the static flexibility matrix, numbers 7 and 8, are the least important with respect to the stiffness matrix. This is intuitively reassuring because the stiffness and flexibility matrices are inverses of each other. Unfortunately, the ith-mode contribution to the stiffness matrix error cannot be computed, as was done with the flexibility matrix error. Therefore, a direct comparison between the norm of an ith-mode contribution to the stiffness matrix error and the SWI is not possible.

Bounding Errors in Model Prediction

Because the goal of the model update process is to improve the predictive performance of a given model, it is natural to ask what effect the choice of error residual will have on the quantities predicted by the updated model. In this section, three possible predicted quantities are examined: static displacements, static loads, and mode shapes and frequencies.

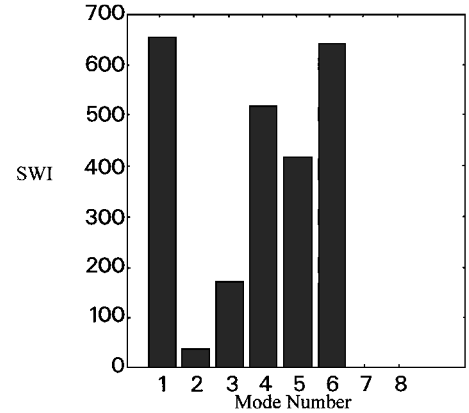

Fig. 4 SWI for mass-normalized modes of Kabe's problem.[7]

Prediction of Static Displacements

Applied static loads are related to the static displacement response of a structure through the flexibility matrix G. This relationship can be expressed as

$$\{q\} = G\{f\} \tag{42}$$

where $\{f\}$ is the vector of applied static loads and $\{q\}$ is a vector of static displacement responses. Given $\{f\}$, the predicted displacement $\{q\} + \{\Delta q\}$ is related to the applied static loads by

$$\{q\} + \{\Delta q\} = (G + \Delta G)\{f\} \tag{43}$$

where $\{\Delta q\}$ and ΔG are the error in displacement response and static flexibility matrix, respectively. Subtracting Eq. (42) from Eq. (43), we get the error in the displacement response as a function of the error in the flexibility matrix and the applied static loads:

$$\{\Delta q\} = \Delta G\{f\} \tag{44}$$

Using consistent vector and matrix norms to quantify the error, we can use the submultiplicative property[6] to bound the error in the displacement response. This gives the following inequality:

$$\|\{\Delta q\}\| \leq \|\Delta G\| \, \|\{f\}\| \tag{45}$$

From Eq. (45), we see that $\|\Delta G\|$ determines an upper bound on the displacement error for any static loading pattern of consistent magnitude. In other words, reducing the static flexibility error directly limits the amount of error in the static displacement response. What then is the effect of minimizing other error residuals? As shown in the preceding analysis, the other error residuals correspond to different weightings on the measured modal data. For example, minimization of a stiffness error residual will generally lead to an update that emphasizes the errors in modes that have large frequency and large modal mass. Similarly, minimization of a modal error residual will simply equally weight the mass-normalized modes. Because these error residuals do not emphasize modes that dominate the static flexibility matrix, one would expect the minimization of them to result in a larger ΔG when model form errors and measurement noise are present. In other words, the error in predicted static displacements will not be bound as tightly, as shown in Eq. (45).

Prediction of Static Forces

We can analyze the prediction of static forces in a manner similar to the analysis of displacement response. This time, the static force response is related to a set of applied displacements through the stiffness matrix, giving the following relationship:

$$\{f\} = K\{q\} \tag{46}$$

Given the applied displacements, the predicted static loads will be given by

$$\{f\} + \{\Delta f\} = (K + \Delta K)\{q\} \tag{47}$$

where $\{\Delta f\}$ and ΔK are the error in the predicted static loads and stiffness matrix, respectively. Subtracting Eq. (46) from Eq. (47) leaves an expression for the static load error in terms of the stiffness matrix error,

$$\{\Delta f\} = \Delta K q \tag{48}$$

Again using consistent vector and matrix norms and invoking the submultiplicative property, we arrive at the following:

$$\|\Delta f\| \leq \|\Delta K\| \|\{q\}\| \tag{49}$$

In this case, $\|\Delta K\|$ determines an upper bound on the force error for any applied displacement vector of the same magnitude. An argument similar to that given earlier may be applied to demonstrate that a stiffness error residual will generally provide the tightest bound on the error in predicted static forces, when model form errors and measurement noise are present.

Prediction of Mode Shapes and Frequencies

If the goal is to minimize the error in predicted frequencies and mode shapes, the modal error residuals $\Delta\Phi$ and $\Delta\Lambda$ may be minimized directly. This can be accomplished by solving a constrained minimization problem as with OMU techniques, by direct comparison of individual mode shapes and frequencies as with modal-based sensitivity methods, or through the use of pole-placement via output feedback as in eigenstructure assignment techniques. All of these techniques try to minimize the errors in predicted mode shapes and frequencies. As already discussed, however, minimizing a modal residual will generally lead to higher errors in a flexibility or stiffness matrix error residual. In other words, a model updated by means of minimizing a modal residual will generally be less accurate in the prediction of static displacement or force responses, unless weighted to approximate a static flexibility or stiffness residual.

Effect of Error Residual Choice in Presence of Model Form Errors

In this section, a modified version of Kabe's problem is used to demonstrate the effect of performing a model update in the presence of model form errors. In this problem, model errors are simulated by adding two masses and four springs to the original Kabe model.[7] The new system is shown in Fig. 5, and values for the added mass and stiffness elements are given in Table 2. For these simulations, the new modified system is used to generate the simulated experimental data. The other exact parameters and the parameters for the nominal model to be updated remain the same as those given in Table 1.

Simulations were conducted to contrast two techniques. The first is the OMU approach developed by Baruch and Bar Itzhack[8] and Baruch.[9–12] This procedure generates a closed-form solution for the updated stiffness matrix. This technique was selected for comparison because it is a modal-based technique, which is easily implemented, it is used extensively in the literature, and it is capable of exactly reproducing the measured modal data. The second set of results is obtained using the linear solution for the PseudoFLEX method presented by Denoyer and Peterson.[2]

In these simulations, it is assumed that the measurements are perfect and all model DOFs are measured. For both techniques, only the first eight mode shapes and frequencies are generated and used as the simulated data. Again, the analytical model assumes only 8 DOFs, instead of 10. With the Baruch/Bar Itzhack method, these data are used directly to update the stiffness matrix. With the PseudoFlex method, the data are first used to generate a measured static flexibility matrix. The measured static flexibility matrix is then used to update the 14 stiffness parameters of the model, via the procedure outlined by Denoyer and Peterson.[2]

After implementing the model update procedures, the updated models are used to predict both static displacements and modal frequencies. Assuming unit static loads applied simultaneously at all eight model DOFs, the static displacements $\{q\}$ are found by solving

$$K\{q\} = \{f\} \tag{50}$$

where $\{f\}$ is a vector of unit loads. The modal frequencies are found by solving the following eigenequation:

$$K\Phi = M\Phi\Lambda \tag{51}$$

Table 3 presents the static displacement response for each of the models at the eight model DOFs, and Table 4 shows the modal frequencies associated with each model. The frequencies are shown relative to the exact modal frequencies with the highest degree of correlation as determined by the modal assurance criterion.[13] The average and maximum errors associated with the predictions of each model are presented in Table 5. These results show that, in the presence of these model form errors, the choice of error residual has a significant impact on the predictive capability of the model. The modal residual-based Baruch/Bar Itzhack method has no error in the predicted frequencies, as expected. However, the model's ability to accurately predict static deflections is severely compromised. On the other hand, the static flexibility residual-based PseudoFLEX method results in a model that is very accurate in the prediction of static displacements. The small error that exists is because there were

Table 2 Additional mass and stiffness parameters for modified Kabe problem

Mass values	Exact stiffness values
$m_9 = 0.010$	$k_{15} = 1.0$
$m_{10} = 0.500$	$k_{16} = 1.0$
——	$k_{17} = 1.0$
——	$k_{18} = 3.0$

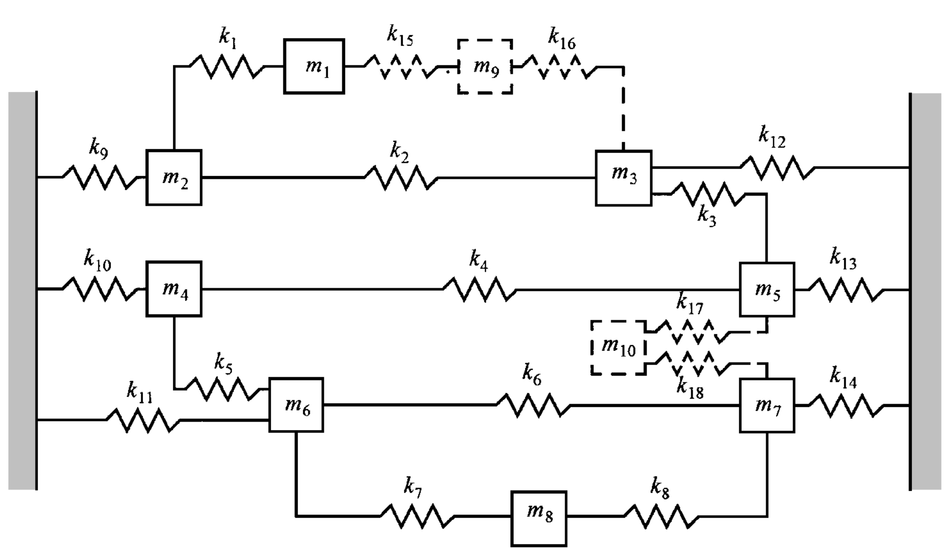

Fig. 5 Kabe's problem[7] with additional model form errors.

Table 3 Applied static forces and static displacement response for modified Kabe update

DOF	Applied force vector	Exact displacement	Nominal model displacement	Ref. 8–12 model displacement	PseudoFLEX model displacement
1	1.0	0.5016	0.5013	6.3574	0.5016
2	1.0	0.0017	0.0013	−0.0023	0.0019
3	1.0	0.0012	0.0008	0.0034	0.0012
4	1.0	0.0012	0.0019	0.0313	0.0012
5	1.0	0.0011	0.0018	−0.0071	0.0011
6	1.0	0.0015	0.0012	−0.1932	0.0015
7	1.0	0.0014	0.0009	−0.1324	0.0014
8	1.0	0.2872	0.1678	61.8840	0.2872

Table 4 Predicted modal frequencies for modified Kabe update

Mode number	Exact modal frequencies	Nominal model frequencies	Ref. 8–12 model frequencies	PseudoFLEX model frequencies
1	0.4496	——	0.4496	——
2	1.9952	——	1.9952	——
3	4.8819	3.9422	4.8819	4.8312
4	5.0546	5.0181	5.0546	5.0296
5	5.0627	6.1714	5.0627	5.0668
6	5.1522	6.1847	5.1522	5.3180
7	5.4433	6.6500	5.4433	5.5235
8	5.6607	6.6869	5.6607	5.6699
9	6.6668	7.1320	——	6.6668
10	8.0277	8.7283	——	7.1275

Table 5 Comparison of static deflection and frequency errors for modified Kabe problem

Error	Nominal model, %	Ref. 8–12 model, %	PseudoFLEX model, %
Average abs. static disp.	35.6	6,055.3	1.2
Maximum abs. static disp.	63.8	21,447.2	6.3
Average abs. frequency	14.7	0.0	2.2
Maximum abs. frequency	22.2	0.0	11.2

two residual modes that were neglected in forming the measured flexibility matrix inasmuch as only the first eight were used. As predicted by the preceding analysis, the PseudoFLEX update fails to minimize the modal error. However, when the modes from the updated model are correlated to the experimental modes, we see that the prediction of modal frequencies is still improved over the nominal model.

One interesting aspect of static flexibility-based update methods such as PseudoFLEX is that, by weighting data to emphasize static behavior only, the approach effectively filters out inertia errors when updating stiffness parameters. Logic suggests that this should lead to better estimates for these parameters. However, this approach does nothing to correct the inertia properties of the model. If the model is to be used to predict quantities that depend heavily on inertia properties, e.g., acceleration response, we would expect this approach to yield poor results. However, the current work suggests a possible two-step approach for dealing with this problem. One could perform an initial update using a static flexibility approach for updating stiffness parameters. A subsequent update (possibly using a modal residual) could then be used to correct the inertia properties. Whereas not investigated in the current work, this will be the subject of future research.

Conclusion

Analytical expressions are presented that relate errors in dynamically measured static flexibility matrices, stiffness matrices, and modal matrices. It has been demonstrated that static flexibility, stiffness, and modal error residuals correspond to unique weightings on the measured modal data. Two indicators, the FWI and the SWI, have been developed to provide a simple way of ranking a set of measured modes according to their potential contribution to static flexibility or stiffness matrix errors. The results demonstrate that these indicators provide a better relative measure of the actual contribution of the ith mode to flexibility or stiffness errors than a simple frequency weighting.

Also examined is the effect of the error residual choice on the ability of the updated model to accurately predict static displacements, static loads, and mode shapes and frequencies. The analysis shows that minimization of a static flexibility error residual results in the least uncertainty in predicted static displacements. Similarly, minimization of a stiffness error residual results in the least uncertainty in predicted static loads, and minimization of a modal error residual results in the least uncertainty in predicted mode shapes and frequencies. Numerical simulation results show that this accuracy is obtained at the expense of accuracy in the other predicted quantities, when the model contains model form errors. Although not investigated, a two-step process is suggested for updating both the stiffness and inertia properties of a model, which may lead to superior results when predicted quantities are sensitive to both stiffness and inertia errors.

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